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$q \leftrightarrow q^{-1}$ invariance of q-oscillators and new realizations of quantum algebras

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Abstract. New realizations of quantum algebras $\mathrm{su}_q(2)$ and $\mathrm{su}_q(1,1)$ are constructed in terms of the q-bosonic and the q-fermionic oscillators. They satisfy the defining commutation relations of quantum algebras without giving any new structure on the q-oscillators. A new relation between the q-oscillators defined for q and those for q^{-1} is found. It converts the defining relations of the q-oscillators for q^{-1} to those for q. Using this relation, it can be shown that our realizations are invariant under $q \leftrightarrow q^{-1}$.

1. Introduction

In recent times quantum algebras [1] have attracted special interest in many fields of both physics and mathematics. Their representation theories are highly developed by the introduction of the q-bosonic and the q-fermionic oscillator realizations of quantum algebras [2–5]. q-oscillators enable us to represent quantum algebras on the Fock space and they are also the powerful tools to construct the q-deformed superalgebras [6] and the q-deformed Virasoro algebra [7]. The q-bosonic oscillator has a richer structure than the ordinary bosonic oscillator, so it is interesting to investigate itself. Some consideration based on the Yang–Baxter equation and q-series have been given [8].

In this paper, we construct new realizations of quantum algebras in terms of the q-bosonic and q-fermionic oscillator. These realizations satisfy the defining commutation relations of quantum algebras as operator identity without giving new structure on the q-oscillators. We further investigate a relation between q-oscillators defined for a value of q and those for its inverse q^{-1}.

To make clear our assertion, let us summarize the present day status of q-oscillator realization taking the $\mathrm{su}_q(2)$ as an example. The q-bosonic oscillator algebra $\mathcal{A}(q)$ is generated by the q-annihilation and q-creation operators a, $a^\dagger$ and the number operator N which satisfy

$$[N, a^\dagger] = a^\dagger \qquad [N, a] = -a \qquad aa^\dagger - qa^\dagger a = q^{-N}. \tag{1.1}$$

To introduce the operation * in $\mathcal{A}(q)$ we suppose $q \in \mathbb{R}$ (which we do throughout this paper suppose)

$$(a)^* = a^\dagger \qquad (a^\dagger)^* = a \qquad N^* = N. \tag{1.2}$$

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The state vectors can be constructed as usual, supposing the existence of a q-vacuum ket such as

$$a|0\rangle = 0 \qquad N|0\rangle = 0 \tag{1.3}$$

the orthonormal n-quanta eigenstate $\{|n\rangle\}$ are given by

$$|n\rangle = ([n]!)^{-1/2}(a^\dagger)^n|0\rangle \tag{1.4}$$

where $[n] \equiv (q^n - q^{-n})/(q - q^{-1})$ and $[n]! = [n][n-1]\cdots[1]$.

Similar to the Jordan–Schwinger realization of su(2) [9], we introduce two independent (mutually commuting) q-bosonic oscillators $a_i^\dagger$, a_i, N_i, $i = 1$, 2 and construct the operators

$$J_+ = a_1^\dagger a_2 \qquad J_- = a_2^\dagger a_1 \qquad J_z = \tfrac{1}{2}(N_1 - N_2). \tag{1.5}$$

The eigenstates $|j\ m\rangle$ of J_z are q-analogue of the angular momentum states

$$|j\ m\rangle = ([j+m]![j-m]!)^{-1/2}(a_1^\dagger)^{j+m}(a_2^\dagger)^{j-m}|0\rangle. \tag{1.6}$$

A peculiarity of this realization is that the operators of (1.5) *do not* satisfy the defining commutaion relations of $su_q(2)$

$$[J_z, J_\pm] = \pm J_\pm \qquad [J_+, J_-] = [2J_z] \tag{1.7}$$

as *operator identity*, but these defining commutation relations *hold* when the operators of (1.5) act on the eigenstates $|j\ m\rangle$. This situation is similar to Dirac's quantization theory of constrained systems.

To make the defining commutation relations of $su_q(2)$ hold as operator identity, we must give additional structure to the q-bosonic oscillator

$$a^\dagger a = [N]. \tag{1.8}$$

This relation holds on the eigenstates of (1.4), which is the reason why the defining relations of $su_q(2)$ hold on the eigenstates $|j\ m\rangle$. On the other hand, (1.8) is closely connected with the invariance of the q-oscillator under $q \leftrightarrow q^{-1}$ when it is regarded as operator relation. This relation can be derived if we identify the generators of the algebra $\mathcal{A}(q)$ and those of $\mathcal{A}(q^{-1})$ [7,10] i.e.

$$\bar{a} = a \qquad \bar{N} = N \tag{1.9}$$

where $\bar{a}, \bar{N} \in \mathcal{A}(q^{-1})$ and $a, N \in \mathcal{A}(q)$. This can be regarded as an expression of $q \leftrightarrow q^{-1}$ invariance of q-oscillator. It is, however, possible to set another relation for the generators of $\mathcal{A}(q)$ and $\mathcal{A}(q^{-1})$, which is also able to regard as an expression of $q \leftrightarrow q^{-1}$ invariance of q-oscillator as seen in section 3.

In the next section, new realizations of $su_q(2)$ and $su_q(1,1)$ are constructed. They satisfy the defining commutaion relations of quantum algebras as operator identity without assuming the relation (1.9) (i.e. (1.8)). In section 3, the invariance of q-oscillator under $q \leftrightarrow q^{-1}$ is discussed. In section 4, the case of q-fermionic oscillators is considered. A new realization of $su_q(2)$ in terms of q-fermionic oscillators is constructed and the invariance of q-fermionic oscillator under $q \leftrightarrow q^{-1}$ is discussed. Section 5 is a summary.

2. q-bosonic oscillator realization of $su_q(2)$ and $su_q(1,1)$

By the defining relations of the algebra $\mathcal{A}(q)$, there exists a central element for $\mathcal{A}(q)$ [11]

$$C = q^{-N}([N] - a^\dagger a) \qquad [C, N] = [C, a^\dagger] = [C, a] = 0. \tag{2.1}$$

This allows us to express $a^\dagger a$ and $aa^\dagger$ in terms of the operators N and C

$$a^\dagger a = [N] - q^N C \qquad aa^\dagger = [N+1] - q^{N+1} C. \tag{2.2}$$

It should be noted that these are the direct consequence of the relation (1.1). Then the operators

$$\begin{aligned} J_+ &= (F_1 F_2)^{-1/4} a_1^\dagger a_2 \qquad J_- = (F_1 F_2)^{-1/4} a_2^\dagger a_1 \\ J_z &= \frac{1}{2}\left(N_1 - N_2 + \frac{\ln\sqrt{F_1}}{\ln q} - \frac{\ln\sqrt{F_2}}{\ln q}\right) \end{aligned} \tag{2.3}$$

where

$$F_i = 1 - (q - q^{-1}) C_i \qquad i = 1,\ 2$$

satisfy the defining commutation relations of $su_q(2)$ as operator identity. It is easily verified by making use of the fact that (2.2) can be rewritten as

$$a^\dagger a = \sqrt{F}\left[N + \frac{\ln\sqrt{F}}{\ln q}\right] \qquad aa^\dagger = \sqrt{F}\left[N + \frac{\ln\sqrt{F}}{\ln q} + 1\right].$$

To get the matrix representation of $su_q(2)$ on the q-bosonic Fock space, we suppose the existence of the following q-vacuum ket:

$$a_i|0\rangle = 0 \qquad N_i|0\rangle = c_i|0\rangle \qquad i = 1, 2. \tag{2.4}$$

The second relation guarantees the non-trivial eigenvalue of the central element (i.e. $C \neq 0$). It also implies that in the limit of $q \to 1$ the q-bosonic oscillator a, $a^\dagger$ and the number operator N are reduced to α, $\alpha^\dagger$ and $\alpha^\dagger\alpha + c_i$, respectively, where α, $\alpha^\dagger$ denote the ordinary bosonic oscillator. Our realization of $su_q(2)$, therefore, is reduced to the Jordan–Schwinger realization of su(2) in the limit of $q \to 1$. The orthonormal eigenvectors of the number operators N_i are given by

$$|n\rangle_i = (q^{-nc_i}[n]!)^{-1/2} (a_i^\dagger)^n |0\rangle \tag{2.5}$$

and when the generators of $\mathcal{A}(q)$ act on these states, they give

$$\begin{aligned} &N_i|n\rangle_i = (n + c_i)|n\rangle_i \\ &a_i^\dagger|n\rangle_i = \sqrt{q^{-c_i}[n+1]}\,|n+1\rangle_i \qquad a_i|n\rangle_i = \sqrt{q^{-c_i}[n]}\,|n-1\rangle_i \end{aligned} \tag{2.6}$$

so that the eigenvalue of the central element is

$$C_i|n\rangle_i = q^{-c_i}[c_i]|n\rangle_i. \tag{2.7}$$

This representation of the algebra $\mathcal{A}(q)$ is not invariant under $q \leftrightarrow q^{-1}$.

The eigenvectors of J_z are given by $|j\ m\rangle = |j+m\rangle_1 |j-m\rangle_2$, which give the correct representation of $\mathrm{su}_q(2)$

$$J_z|j\ m\rangle = m|j\ m\rangle \qquad J_{\pm}|j\ m\rangle = \sqrt{[j \mp m][j \pm m + 1]}\,|j\ m \pm 1\rangle.$$

It is remarkable that the $\mathrm{su}_q(2)$ representation is invariant under $q \leftrightarrow q^{-1}$ and do not contain the constants c_i $(i = 1,\ 2)$ in spite of the representation of $\mathcal{A}(q)$ mentioned above.

Let us turn to the quantum algebra $\mathrm{su}_q(1,1)$ [10,12]. The defining commutation relations of $\mathrm{su}_q(1,1)$ are given by

$$[K_z, K_{\pm}] = \pm K_{\pm} \qquad [K_+, K_-] = -[2K_z] \tag{2.8}$$

and the central element is given by

$$C[su_q(1,1)] = [K_z][K_z+1] - K_-K_+ = [K_z][K_z-1] - K_+K_-. \tag{2.9}$$

A representation, which is reduced to the positive discrete series of su(1,1) [13] in the limit of $q \to 1$, is given as

$$\begin{aligned} &C[su_q(1,1)]|\kappa\ \mu\rangle = [\kappa][\kappa-1]|\kappa\ \mu\rangle \\ &K_z|\kappa\ \mu\rangle = \mu|\kappa\ \mu\rangle \\ &K_{\pm}|\kappa\ \mu\rangle = \sqrt{[\mu \pm \kappa][\mu \mp \kappa \pm 1]}\,|\kappa\ \mu \pm 1\rangle. \end{aligned} \tag{2.10}$$

In this representation, the possible values of μ are given by $\mu = \kappa,\ \kappa+1,\ \kappa+2, \ldots$ to infinity, while κ may take any positive real number.

We give two kinds of realizations of $\mathrm{su}_q(1,1)$ in terms of the q-bosonic oscillator, one of them consists of two indepent q-bosonic oscillators and the other consists of one kind of q-bosonic oscillator. The latter realization is somewhat more complicated than the former one. Both realizations give representations which are reduced to the positive discrete series of su(1,1) in the limit of $q \to 1$.

Let us first consider the case of two independent q-oscillators. The $\mathrm{su}_q(1,1)$ generators are constructed as

$$\begin{aligned} &K_+ = (F_1F_2)^{-1/4} a_1^{\dagger} a_2^{\dagger} \qquad K_- = (K_+)^* \\ &K_z = \frac{1}{2}\left(N_1 + N_2 + \frac{\ln\sqrt{F_1}}{\ln q} + \frac{\ln\sqrt{F_2}}{\ln q} + 1\right). \end{aligned} \tag{2.11}$$

Its eigenstates are given by

$$|\kappa = \tfrac{1}{2}(|n_1 - n_2| + 1) \qquad \mu = \tfrac{1}{2}(n_1 + n_2 + 1)\rangle = |n_1\rangle_1|n_2\rangle_2. \tag{2.12}$$

As in the case of $\mathrm{su}_q(2)$, the constants c_i do not appear in the representation.

Next, the case of one kind of q-bosonic oscillator is considered. The $\mathrm{su}_q(2)$ generators are constructed as

$$\begin{aligned} &K_+ = a^{\dagger}\beta(N,C)a^{\dagger}\beta(N,C) \qquad K_- = (K_+)^* \\ &K_z = \frac{1}{2}\left(N + \frac{\ln\sqrt{F}}{\ln q} + \frac{1}{2}\right) \end{aligned} \tag{2.13a}$$

where

$$\beta(N,C) = \left\{ \frac{1}{\sqrt{F}} \frac{[\,\frac{1}{2}(N+1+\ln\sqrt{F}/\ln q)\,]}{[\,N+1+\ln\sqrt{F}/\ln q\,]} \right\}^{1/2} . \tag{2.13b}$$

The eigenstates of K_z are the n-quanta states $\{|n\rangle\}$ themselves, but the states with even value of n or odd value of n give different representation of $\mathrm{su}_q(1,1)$, as in the case of $q \to 1$ limit [14]. The states with even n ($n = 2m,\ m = 0,\ 1,\ 2, \ldots$) carry the $\mathrm{su}_q(1,1)$ quantum numbers $\kappa = \frac{1}{4}$ and $\mu = \frac{1}{4} + m$, those with odd n ($n = 2m+1,\ m = 0,\ 1,\ 2, \ldots$) carry the $\mathrm{su}_q(1,1)$ quantum numbers $\kappa = \frac{3}{4}$ and $\mu = \frac{3}{4} + m$.

3. $q \leftrightarrow q^{-1}$ invariance of the q-bosonic oscillator

If we suppose the relation of (1.9) for the elements of the algebra $\mathcal{A}(q)$ and those of $\mathcal{A}(q^{-1})$, the central elements of $\mathcal{A}(q)$ and $\mathcal{A}(q^{-1})$ vanish identically. Keeping the central element non-vanishing, can we set any relation between the elements of $\mathcal{A}(q)$ and $\mathcal{A}(q^{-1})$? If it is possible, what does the term '$q \leftrightarrow q^{-1}$ invariance of q-oscillator' mean?

To answer both questions, we consider the following relations between the elements $a,\ a^\dagger,\ N$ of $\mathcal{A}(q)$ and $\bar{a},\ \bar{a}^\dagger,\ \bar{N}$ of $\mathcal{A}(q^{-1})$

$$\bar{N} = N \qquad \bar{a} = \frac{1}{\sqrt{F}} a \qquad \bar{a}^\dagger = \frac{1}{\sqrt{F}} a^\dagger . \tag{3.1}$$

It is easily verified by using these relations that the defining relations of the algebra $\mathcal{A}(q^{-1})$

$$[\bar{N}, \bar{a}^\dagger] = \bar{a}^\dagger \qquad [\bar{N}, \bar{a}] = -\bar{a} \qquad \bar{a}\bar{a}^\dagger - q^{-1}\bar{a}^\dagger\bar{a} = q^{\bar{N}} \tag{3.2}$$

are reduced to those of the algebra $\mathcal{A}(q)$, (1.1). In this sense, we call the q-bosonic oscillators which satisfy (3.1) the q-bosonic oscillators that are invariant under $q \leftrightarrow q^{-1}$. It is also easily verified that the central element $\bar{C}$ of $\mathcal{A}(q^{-1})$ can be rewritten in terms of that of $\mathcal{A}(q)$ as

$$\bar{C} = F^{-1} C. \tag{3.3}$$

By making use of (3.1) and (3.3), it can be shown that all generators of $\mathrm{su}_q(2)$ and $\mathrm{su}_q(1,1)$ constructed in the preceding section are invariant under $q \leftrightarrow q^{-1}$.

It should be noted that the discussion of this section is not restricted to q being real, it is applicable to arbitrary value of $q \in \mathbb{C}$.

4. The case of the q-fermionic oscillator

For the q-fermionic oscillator algebra $\mathcal{B}(q)$, the similar discussion to the preceding sections is available. The algebra $\mathcal{B}(q)$ is generated by the q-fermionic annihilation and creation operators $b,\ b^\dagger$ and the number operator M which satisfy [6]

$$[M, b^\dagger] = b^\dagger \qquad [M, b] = -b \qquad bb^\dagger + qb^\dagger b = q^M \tag{4.1}$$

and $(b)^2 = (b^\dagger)^2 = 0$. There is a central element

$$D = q^M([M] - b^\dagger b) \qquad [D, M] = [D, b^\dagger] = [D, b] = 0. \tag{4.2}$$

By this definition, the operators $b^\dagger b$ and $bb^\dagger$ can be expressed in terms of the operators M and D

$$b^\dagger b = [M] - q^{-M} D \qquad bb^\dagger = -[M-1] + q^{-M+1} D. \tag{4.3}$$

Let us first construct a realization of $su_q(2)$ by introducing two independent q-fermionic oscillators where by the term 'independent' we mean that

$$\begin{aligned} &\{b_i, b_j\} = \{b_i, b_j^\dagger\} = 0 \\ &[M_i, b_j] = [M_i, b_j^\dagger] = [M_i, M_j] = 0 \qquad \text{for } i \neq j. \end{aligned} \tag{4.4}$$

Then the operators

$$\begin{aligned} &J_+ = (G_1 G_2)^{-1/4} b_1^\dagger b_2^\dagger \qquad J_- = (G_1 G_2)^{-1/4} b_2 b_1 \\ &J_z = \frac{1}{2}\left(M_1 + M_2 - \frac{\ln\sqrt{G_1}}{\ln q} - \frac{\ln\sqrt{G_2}}{\ln q} - 1\right) \end{aligned} \tag{4.5}$$

where

$$G_i = 1 + (q - q^{-1}) D_i \qquad i = 1,\ 2$$

satisfy the defining commutation relations of $su_q(2)$ as operator identity. It should be noted that we use only the defining relation of the algebra $\mathcal{B}(q)$ so far.

Next, the invariance of the q-fermionic oscillator under $q \leftrightarrow q^{-1}$ is considered. One possibility to relate the elements of $\mathcal{B}(q^{-1})$ to those of $\mathcal{B}(q)$ is that

$$\bar{b} = b \qquad \bar{b}^\dagger = b^\dagger \qquad \bar{M} = M$$

where $b,\ b^\dagger,\ M \in \mathcal{B}(q)$ and $\bar{b},\ \bar{b}^\dagger,\ \bar{M} \in \mathcal{B}(q^{-1})$. With the aid of the defining relations of $\mathcal{B}(q)$ and $\mathcal{B}(q^{-1})$

$$b^\dagger b = [M] \qquad D = 0$$

are derived.

As for the case of the q-bosonic oscillator, a more general relation, which keeps the central element non-vanishing, is possible

$$\bar{M} = M \qquad \bar{b} = \frac{1}{\sqrt{G}} b \qquad \bar{b}^\dagger = \frac{1}{\sqrt{G}} b^\dagger. \tag{4.6}$$

It can be shown that the defining relations of the algebra $\mathcal{B}(q^{-1})$ are reduced to those of $\mathcal{B}(q)$ by making use of (4.6). The central element $\bar{D}$ of $\mathcal{B}(q^{-1})$ is expressed by that of $\mathcal{B}(q)$ as $\bar{D} = G^{-1} D$. It is also verified that the realization of $su_q(2)$, (4.5), is invariant under $q \leftrightarrow q^{-1}$.

5. Summary and discussion

We have constructed new realizations of the quantum algebras $\mathrm{su}_q(2)$ and $\mathrm{su}_q(1,1)$ in terms of the q-bosonic and q-fermionic oscillators. They satisfy the defining commutation relations of quantum algebras as operator identity without giving any new structure on q-oscillators, i.e. $a^\dagger a = [N]$. It may be possible to construct the realizations of other quantum algebras, such as $\mathrm{su}_q(n)$, $\mathrm{so}_q(n)$, $\mathrm{sp}_q(n)$ and quantum superalgebras and so on, in the similar manner. The central elements of q-oscillator algebras play a crucial role in our realization. These central elements are reduced to c-numbers in the limit of $q \to 1$, which is consistent with the fact that there are no central operators in both the bosonic and the fermionic oscillator algebras. It is hopeless to regard the central elements of the q-oscillator algebras as a central elements of the q-deformed Virasoro algebra in a single mode realization such as [7], because it is well-known that the central element do not appear in the realization of the Virasoro algebra by one kind of bosonic oscillator and this property must be preserved in the limit of $q \to 1$ when a q-deformaton of the Virasoro algebra is discussed.

It is not necessary to require $q \in \mathbb{R}$ when relations between operators are discussed. Our realizations are valid for arbitrary value of $q \in \mathbb{C}$. The norm of eigenstates, however, are not able to calculate for $q \in \mathbb{C}$ and $|q| \neq 1$, since there is no relation between $a(q)$ and $a(q^*)$. It seems to be a challenging problem to relate $a(q)$ to $a(q^*)$ in order for further understanding of the q-oscillators.

The other consequence of this paper is the generalization of the notion of $q \leftrightarrow q^{-1}$ invariance of the q-oscillators. A new relation between the elements of $\mathcal{A}(q)$ (or $\mathcal{B}(q)$) and those of $\mathcal{A}(q^{-1})$ (or $\mathcal{B}(q^{-1})$) have been introduced. They reduce the defining relations of the algebra $\mathcal{A}(q^{-1})$ (or $\mathcal{B}(q^{-1})$) to those of $\mathcal{A}(q)$ (or $\mathcal{B}(q)$). In this sense, we call the q-oscillators invariant under $q \leftrightarrow q^{-1}$. As a consequence of this invariance of the q-oscillators under $q \leftrightarrow q^{-1}$, our realizations of quantum algebras are invariant under $q \leftrightarrow q^{-1}$.

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